# A Proof of Seymour's Second Neighborhood Conjecture

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## ABSTRACT:

This paper verifies Seymour's Second Neighborhood Conjecture, which states that for every oriented digraph there exists at least one vertex $v$ such that $v$ has at least as many neighbors at distance two as it has at distance one. The verification of this conjecture in turn implies partial results for the Caccetta-Häggkvist Conjecture and the Behzad-Chartrand-Wall Conjecture.

# 1 INTRODUCTION

A *digraph* $\Gamma$ is a set of vertices $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and an arc set $R(\Gamma)$ composed of ordered pairs of these vertices. If $(v_i, v_j) \in R(\Gamma)$ then we say that $v_i$ points towards $v_j$. An oriented digraph is one in which there are no loops (that is, $(v_i, v_i) \notin R(\Gamma)$ for all $i$) and there are no digons (that is, if $(v_i, v_j) \in R(\Gamma)$ then $(v_j, v_i) \notin R(\Gamma)$ for all $i, j$). For an oriented digraph $\Gamma$, let the outdegree of $v_0$, denoted $od_\Gamma(v_0)$, be the number of vertices $v_0$ points towards. Similarly, let the indegree of $v_0$, denoted $id_\Gamma(v_0)$, be the number of vertices which point toward $v_0$. Finally, let $d_i(v, \Gamma)$ be the number of vertices at distance $i$ from $v$ in the oriented digraph $\Gamma$.

Define the square of an oriented digraph $\Gamma$, denoted $\Gamma^2$, to be the digraph on the vertex set $V(\Gamma)$ with the arc set $R^2(\Gamma) = R(\Gamma) \cup S$ where $(u, v) \in S$ if and only if $(u, w), (w, v) \in R(\Gamma)$ and $(u, v) \notin R(\Gamma)$.

*Seymour's Second Neighborhood Conjecture (SSNC)* (Fisher quoting [3]) states:

For every oriented digraph $\Gamma$ there exists a $v$ such that $d_1(v, \Gamma^2) \leq 2d_1(v, \Gamma)$.

This can be equivalently stated as [2]:

For every oriented digraph, there exists a $v$ such that $d_1(v) \leq d_2(v)$.

Fisher proved SSNC for tournaments (also known as Dean's Conjecture) [4]. We provide a proof of this conjecture for all oriented digraphs.

The truth of this theorem implies two other partial results:

(i) *Behzad-Chartrand-Wall Conjecture:* Define a *directed cage* to be the smallest *d-regular* digraph, a digraph with $id(v_i) = od(v_i) = d$ for all $i$, of girth $g$, that is having shortest directed cycle of length $g$. The conjecture states that the number of vertices $n$ for a directed cage is:

$$n = (g-1)d + 1$$

SSNC implies the $d = \lceil \frac{n}{3} \rceil$ case of this conjecture [2].

(ii) *Caccetta-Häggkvist Conjecture:* This conjectures states that given $\Gamma$, a digraph on $n$ vertices in which $od(v_i) \geq d$ for all $i$, the girth of $\Gamma$ is at most $\lfloor \frac{n}{d} \rfloor$. SSNC implies the interesting case of $d = \frac{n}{3}$ [5].

*Note:* As much as possible, graph theory terminology is taken from [1].

## 2 DEFINITIONS

**Definition** An oriented digraph (a loopless, directed graph without digons) is said to satisfy Seymour's Second Neighborhood Conjecture at a vertex $v_0$ iff $d_1(v_0) \leq d_2(v_0)$.

**Definition** Given an oriented tree, a *leaf* is any vertex $u$ such that $od(u) + id(u) = 1$.

**Definition** A *directed cycle* is a cycle with all arcs either pointing clockwise or anti-clockwise.

**Definition** A non-uniform cycle $C_n^*$ on $n$ vertices is a cycle that is not oriented clockwise or anti-clockwise.

**Definition** A $j$ *– complete sun* $C_j^n$ of order $j$ on $n$ vertices is a directed cycle with $j$ arcs, called *rays*, pointing outward from each vertex on the cycle.

## 3 PROOF

**Proposition 1** *Suppose $\Gamma$ has a vertex $v_0$ such that $od_\Gamma(v_0) = 0$. Then $\Gamma$ satisfies SSNC at $v_0$.*

*Proof* Because $od_\Gamma(v_0) = 0$, $v_0$ has no neighbors. Thus $d_1(v_0, \Gamma) = d_2(v_0, \Gamma) = 0$.

□

**Proposition 2** *Suppose $\Gamma$ is an oriented digraph with some subgraph $\Gamma_{v_0}$ containing a vertex $v_0$ such that $\Gamma_{v_0}$ satisfies SSNC at $v_0$ and $od_\Gamma(v_0) = od_{\Gamma_{v_0}}(v_0$ then $\Gamma$ satisfies SSNC at $v_0$.*

*Proof* Because $od_\Gamma(v_0) = od_{\Gamma_{v_0}}(v_0)$, the number of vertices $v_0$ points towards is equal in both graphs. Therefore, $d_1(v_0, \Gamma) = d_1(v_0, \Gamma_{v_0})$. Since $\Gamma_{v_0}$ is a subgraph of $\Gamma$, $d_2(v_0, \Gamma_{v_0}) \leq d_2(v_0, \Gamma)$. The proposition then follows.

□

**Lemma 1** *In every oriented tree $\Gamma$, there exists at least one vertex $v_0$ such that $od_\Gamma(v_0) = 0$.*

*Proof* Consider the leaves of a tree. If any of the leaves is a sink, then it has outdegree zero and Lemma 1 is satisfied. Suppose then that all leaves are sources. Remove all these leaves and the associated arcs. This creates a subtree. If any of the leaves of this subtree is a sink, it has outdegree zero because all removed leaves and arcs only contribute to the indegree of the leaves of the subtree. If all these new leaves are sources, we may repeat the process by removing leaves and the associated arcs from them. This process must terminate at some point because the graph is finite. Therefore, there must be some leaf in a constructed subtree that is a sink and, because all removed leaves and arcs only contribute to its indegree, has outdegree zero.

□

**Lemma 2** *Any digraph containing at least one oriented cycle satisfies SSNC for at least one vertex on that cycle.*

*Proof* We will first prove j-complete suns satisfy SSNC at every vertex on the graph. First note that the sink of every ray of a j-complete sun has outdegree 0, and, thus, the sun satisfies SSNC at these vertices. It then suffices to consider only vertices on the cycle. This will be done by induction on j.

For $j = 0$, $C_n^0$ is a directed cycle which obviously satisfies SSNC at every point on the cycle. Assume $C_n^{j-1}$ satisfies SSNC at every point. Now consider a vertex $v_i$ on the cycle. This vertex has one more neighbor at distance one in $C_n^j$ than in $C_n^{j-1}$, namely the sink of its additional ray. The vertex $v_{i+1}$ on the cycle which $v_i$ points towards also has one additional neighbor at distance one on its ray, call it $u$. Since $v_i$ points towards $v_{i+1}$ and $v_{i+1}$ points towards $u$, $u$ is at distance two from $v_i$. Thus $d_1(v_i) \leq d_2(v_i)$. Since $v_i$ is arbitrary, this inequality holds for all vertices on the cycle.

We will now prove Lemma 2 for any digraph $\Gamma$ containing a directed cycle. $\Gamma$ has a j-complete sun subgraph with $j \geq 0$. Select such a subgraph for which $j$ is the maximum possible. Because $C_n^j$ is maximal, for some $v_0$ on the cycle $od_\Gamma(v_0) = od_{C_j^n}(v_0) = j + 1$. If this were not the case, and $od_\Gamma(v_0) > j+1 \ \forall \ v$ on the cycle, then there would exist a subgraph $C_n^k$ where $k > j$. By Proposition 2, $\Gamma$ satisfies SSNC at this vertex $v_0$.

**Theorem** *Every oriented digraph $\Gamma$ satisfies SSNC.*

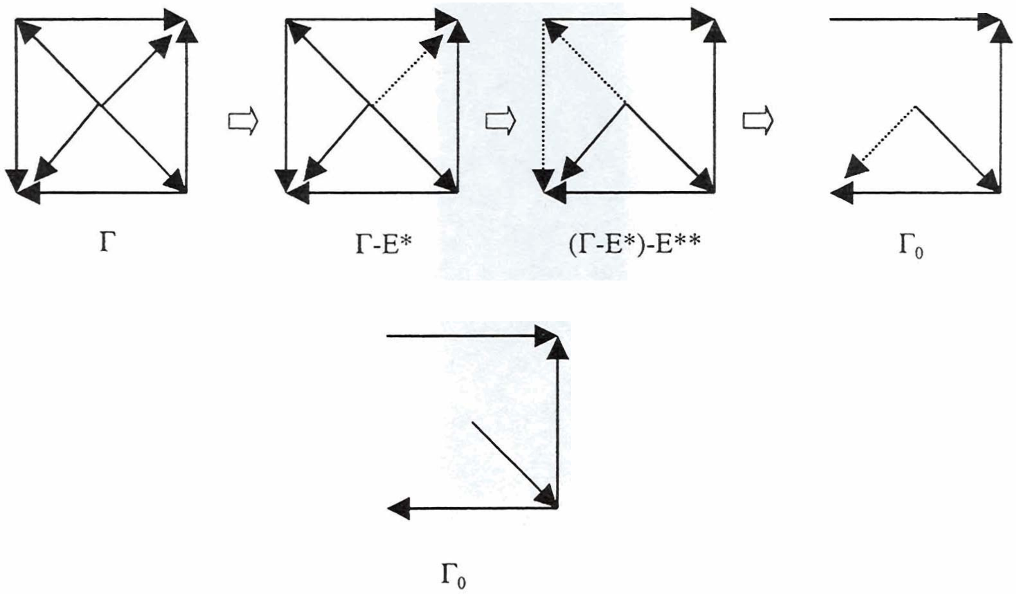
*Proof* By Proposition 1 and Lemma 1, if $\Gamma$ is a tree, $\Gamma$ satisfies SSNC. By Lemma 2, if $\Gamma$ contains a directed cycle, it satisfies SSNC. Therefore, it suffices to assume $\Gamma$ contains cycles, none of which is directed. That is, $\Gamma$ contains at least one non-uniform cycle and no directed cycles. Choose a direction (either clockwise or anti-clockwise). (Without loss of generality, choose anti-clockwise). Then consider a cycle in $\Gamma$. For this cycle, remove all arcs oriented anti-clockwise and any now isolated vertices. If the resulting subgraph still has cycles, choose one of these and again remove all arcs oriented anti-clockwise and any isolated vertices. Repeat this procedure until the resulting graph $\Gamma_0$ has no cycles. This graph $\Gamma_0$ must have at least one arc. If it does not, it would imply the removal of the anti-clockwise arcs in the last cycle removed all arcs in the graph. This would imply the last cycle was directed anti-clockwise, which contradicts our assumptions on $\Gamma$. Since $\Gamma_0$ has no cycles, it is a forest. By Lemma 1, some vertex $v_0$ of $\Gamma_0$ has outdegree equal t zero. We will show that the addition of the erased arcs will not alter the outdegree of this vertex.

If $v_0$ does not belong to any cycle, we are done. Assume $v_0$ belongs to some cycle. All arcs that were removed were anti-clockwise with respect to some cycle. Thus, arcs that remain are clockwise with respect to some cycle. Let vertex $u$ be a vertex in $\Gamma_0$ to which we must add an erased anti-clockwise arc $u\vec{w}$. Addition of an anti-clockwise arc to a clockwise arc in the same cycle will necessarily match the sources of these two arcs. Thus $u$ is also the source of some existing clockwise edge in $\Gamma_0$. Consequently, $od_{\Gamma_0}(u) > 0$ and the vertex $u$ can not be our $v_0$. This is true at each stage in which we add back erased arcs for all the sources of these arcs. Thus, the addition of removed arcs will not alter the outdegree of $v_0$.

By Proposition 1, $\Gamma$ satisfies SSNC at this $v_0$.

□

Figure 2: Removing anti-clockwise edges from an oriented digraph $\Gamma$

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